Dupuy, presented to Hermite in the name of the President of the Republic the insignia of Grand Officer of the Legion of Honor, and the messages were read of those who from various parts of the world associated themselves with the splendid ceremony.

High testimony of admiration and sympathy was offered the great geometer more recently upon the occasion of the meeting at Paris, last August, of the international congress of mathematicians. The Congress sent him a telegram of admiration and sympathy (he was at Saint-Jean-de Luz). This act caused vast satisfaction and profound emotion to the scientist, as he wrote me in one of his last letters.

Hermite retained to the last day of his life his privileged intelligence; but his body suffered. In a long letter of his, a few days before his death, he complained of his attacks of asthma and of the lack of appetite and of sleep: he seemed to forsee the nearness of his end, so that sending me one of his works he said that this would be without doubt *the last*! and that he had in great part accomplished it at Saint-Jean-de Luz, whereby the benefit of the mild climate had reawakened his mathematical activity. This last work is a letter to Professor Pincherle published in tomo V of the "*Annali di Matematica.*" He told us also that he had sent a brief article to the new journal "Le Matematiche" of Prof. Alasia.

We will end by expressing a wish We wish that those who have the authority would take the initiative toward an international subscription for a work containing an extended biography of the ever memorable geometer, and a minute analysis of his works; perhaps might be added some brief articles by very illustrious living mathematicians; something, in fine, which would be as a funeral crown offered to the memory of the great dead.

[Written by Juan J. Durán-Loriga for *Le Matematiche*, and translated by the English editor, G. B. Halsted.]

A PROBLEM AND ITS SOLUTION.

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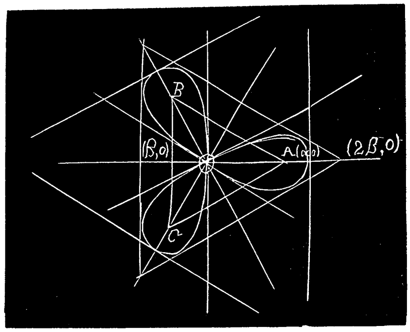
Find that one of these curves $(x^2+y^2)^3=a^3(x^3-3xy^2)$.

I. (a) is the locus of such points that the product of their distances from the vertices of a fixed equilateral triangle is equal to the semi-parameter. (b) The bitangents each touching the curve in two real distinct points are parallel to the sides of the fixed triangle and their six points of contact are on a circle. (c) The sides of the equilateral triangle obtained by the tangents each touching the curve in its three farthest real points from the origin of coördinates are parallel to the tangents of its real triple points. (d) Some other property showing the form of this curve.

II. (a) The "deficiency" (genus) of the curve is $p=1$ and it has three triple points and their tangents are all distinct. (b) Its arc is expressed by the Abelian integral of the first kind attached to its equation (saving a constant). (c) x and y are then two elliptic functions of this arc.

SOLUTION.



I. (a) Let the origin of rectangular coordinates, O, be the center of an equilateral triangle; one of its sides, BC, parallel to y-axis; α, being the arithmetical value of $\frac{\sqrt[3]{a^3}}{2}$ the radius of its circumscribed circle $x^2+y^2=\alpha^2$. Then its vertices are $(x=\alpha,\ y=0)$, $(x=-\frac{1}{2}\alpha,\ y=\frac{\sqrt{3}}{2}\alpha)$, $(x=-\frac{1}{2}\alpha, y=\frac{\sqrt{3}}{2}\alpha)$ (1).

The locus of such points that the products of their distances from the points (1) is $\frac{1}{2}a^3$ is expressed by the equation

$$\sqrt{\{[(x-\alpha)^2+y^2][(x+\tfrac{1}{2}\alpha)^2+(y-\tfrac{\sqrt{3}}{2}\alpha)^2][(x+\tfrac{1}{2}\alpha)^2+(y+\tfrac{\sqrt{3}}{2}\alpha)^2]\}}=\alpha^3.$$

Developing and reducing this equation we get

$$(x^2+y^2)^3=\alpha^3(x^3-3xy^2)\ldots.(2).$$

The sides of the fixed triangle are

$$x+\alpha=0,\quad x-\sqrt{3}y-2\alpha=0,\quad x+\sqrt{3}y-2\alpha=0,\quad (\alpha^3=\tfrac{1}{2}a^3)\ldots.(3).$$

(b) These three straight lines

$$x+\beta=0,\quad x-\sqrt{3}y-2\beta=0,\quad x+\sqrt{3}y-2\beta=0,\quad (\beta^3=\tfrac{1}{4}\alpha^3).\ ..(4),$$

are bitangent to the curve respectively each in one of the following three couples of points:

$$(x=-\beta,\ y=\pm\beta),$$

$$(x=-\frac{\sqrt{3}-1}{2}\beta,\ y=-\frac{\sqrt{3}+1}{2}\beta),\ (x=\frac{\sqrt{3}+1}{2}\beta,\ y=-\frac{\sqrt{3}-1}{2}\beta)..(5).$$

$$(x=-\frac{\sqrt{3}-1}{2}\beta,\ y=\frac{\sqrt{3}+1}{2}\beta),\ (x=\frac{\sqrt{3}+1}{2}\beta,\ y=\frac{\sqrt{3}-1}{2}\beta).$$

The bitangents (4) intersect each other in three points which form the vertices of an equilateral triangle:

$$(x=2\beta,\ y=0),\quad (x=-\beta,\ y=-\sqrt{3}\beta),\quad (x=-\beta,\ y=\sqrt{3}\beta)\ldots.(6),$$

and it is evident that the straights (3) and (4) are three couples of parallels.

All six points (5) stay on the circle $x^2+y^2=2\beta^2$.

(c) The circle $x^2+y^2=a^2$ *touches* the curve in its real farthest points from the origin of coördinates

$$(x=a,\ y=0),\quad (x=-\tfrac{1}{2}a,\ y=\frac{\sqrt{3}}{2}a),\quad (x=-\tfrac{1}{2}a,\ y=-\frac{\sqrt{3}}{2}a)\ldots.(7).$$

Indeed these points are the limits of three couples of real points obtained through intersection with a circle, center O and radius r, variable from 0 to a, for if $r>a$, all twelve points of intersection would be imaginary.

The tangents which touch both this circle and curve are

$$x-a=0,\quad x-\sqrt{3}y+2a=0,\quad x+\sqrt{3}y+2a=0\ldots\ (8).$$

The origin of coördinates is a real triple point of the curve (we will prove later it is alone) for the first and second partial derivatives of (2) vanish for $x=0$, $y=0$; its tangents are three distinct ones

$$x=0,\quad x-\sqrt{3}y=0,\quad x+\sqrt{3}y=0\ldots.(9),$$

which are parallel to the straights (8).

(*d*) The straight lines (3), (4), (8), and (9) give three sets of four parallel and their three directions are inclined 90°, 60°, and 30° to x-axis. The x-axis is an axis of symmetry of the curve for the degree of y in (2) is always a multiple of 2; on this straight lie the first points of (1), (6), and (7). Through two successive rotations (120°) of the coördinate system we get that the straights containing respectively the second and third points (1), (6), and (7) are also axes of symmetry and there is no other one. Then we can say that the curve is a three-lobed one, each branch has finished values and one of them is within the following straights, *i. e.*:

$$x-\sqrt{3}y=0,\quad x+\sqrt{3}y-2\beta=0,\quad x=a,\quad x-\sqrt{3}y-2\beta=0,\quad x+\sqrt{3}y=0,$$

by their points of contact and some other one we can get, it is easy to obtain the form of the curve.

II. (*a*) The curve $x'^3-3x'y'^2=1/a^3$ is the transformed one of (2) through reciprocal vectorial rays and its genus or deficiency is $p=1$; then also the first curve's genus is $p=1$, this transformation being a birational one. The curve (2) has three triple points: the origin of coördinates (a real one) and the two cyclic points $[(1, i, 0)$ and $(1, -i, 0)]$ and no more multiple points for $p=1$. The tangents of the real triple points are expressed by the equations (8) and those of the imaginary points are $(x\pm iy)^3=\tfrac{1}{2}a^3\ z^3$, which are six distinct tangents, of superior inflexion, for each branch is linear and of the third class.

(*b*) The equation of the curves (2), if written in polar coördinates, is

$$\rho^3 = a^3 \cos 3\theta \ldots. (10).$$

The differential expression of its arc s is

$$ds = \frac{a^3 d\theta}{\rho^2} \ldots. (11),$$

or, in Cartesian coördinates,

$$ds = \frac{a^3}{x^2+y^2} d \text{ arc tg} \frac{y}{x},$$

which is a rational function of x and y, and then $\int ds$ is an Abelian integral attached to the equation (2).

From (10) we get $\frac{d\theta}{\rho^2} = \frac{d\rho}{a^3 \sin 3\theta}$, then $s = \int \frac{a^3 d\rho}{\sqrt{(a^6 - \rho^6)}}$, which has a finished value anywhere and thus it is an Abelian integral of the first kind. But the genus of the curve is $p=1$ and it must have only an integral of the first kind attached to its equation, and then, saving a constant, this integral expresses its arc s.

Let us put $\rho^2 = 1/r$; we have

$$s = \int \frac{dr}{\sqrt{[4r^3 - (4/a^6)]}} \ldots (12),$$

which is an elliptic integral of the first kind.

The well known fundamental relation of Weierstrass between his function pu and its first derivative $p'u$, $p'^2 u = 4p^3 u - g_2 pu - g_3$, will be here

$$p'^2 s = 4p^3 s - 4/a^6 \ldots. (13).$$

The coördinates x and y of the curve (2) are expressed by the following functions

$$x = -\frac{3}{a^3} \frac{ps}{p^3 s - (4/a^6)}, \quad y = \frac{\sqrt{-3}}{2} \frac{psp's}{p^3 s - (4/a^6)} \ldots. (14),$$

which are two elliptic functions of the arc s. Indeed eliminating ps and $p's$ between (13) and (14) we get

$$(x^2+y^2)^3 = a^3(x^3 - 3xy^2) = \left(\frac{-3p^2 s}{p^3 s - (4/a^6)}\right)^3.$$

It ought to be so for the coördinates of an elliptic curve are expressed by elliptic functions of the Abelian integral of the first kind relative to its equation.